KNOWLEDGE ORGANISER: Year 13 Core – Chapter 1 – Algebraic Methods

To prove a statement by **contradiction** you start by assuming it is **not true.** You then use logical steps to show that this assumption leads to something impossible, either a contradiction of the assumption or contradiction of a fact you know to be true. You can conclude that your assumption was incorrect, and the original statement **was true**

- A **rational** can be written as $\frac{a}{b}$, where a and b are integers
- A **irrational** cannot be expressed in the form $\frac{a}{b}$, where a and b are integers

Algebraic Fractions

To **multiply fractions**, cancel any common factors, then multiply the numerators and multiply the denominators

$$\frac{x+1}{2} \times \frac{3}{x^2-1} = \frac{x+1}{2} \times \frac{3}{(x+1)(x-1)}$$
$$= \frac{{}^{1}\cancel{x+1}}{2} \times \frac{3}{{}_{1}\cancel{(x+1)}(x-1)}$$
$$= \frac{3}{2(x-1)}$$

To **divide fractions,** multiply the first fraction by the reciprocal of the second fraction

$$\frac{x+2}{x+4} \div \frac{3x+6}{x^2-16} = \frac{x+2}{x+4} \times \frac{x^2-16}{3x+6}$$
$$= \frac{x+2}{x+4} \times \frac{(x+4)(x-4)}{3(x+2)}$$
$$= \frac{\cancel{x+2}^{1}}{\cancel{x+4}^{1}} \times \frac{\cancel{(x+4)}^{1}(x-4)}{3\cancel{(x+2)}^{1}}$$
$$= \frac{x-4}{3}$$

To **add or subtract fractions**, find a common denominator

$$\frac{2}{(x+3)} - \frac{1}{(x+1)} = \frac{2(x+1)}{(x+3)(x+1)} - \frac{1(x+3)}{(x+3)(x+1)}$$
$$= \frac{2(x+1) - 1(x+3)}{(x+3)(x+1)}$$
$$= \frac{2x+2-1x-3}{(x+3)(x+1)}$$
$$= \frac{x-1}{(x+3)(x+1)}$$

Partial Fractions

A single fraction with two distinct linear factors in the denominator can be split into two separate fractions with linear denominators.

$$\frac{\mathbf{5}}{(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}-\mathbf{4})} = \frac{\boldsymbol{A}}{(\boldsymbol{x}+\mathbf{1})} + \frac{\boldsymbol{B}}{(\boldsymbol{x}-\mathbf{4})}$$

The method of partial fractions can also be used when there are more than two distinct linear factors in the denominator:

$$\frac{\mathbf{7}}{(\boldsymbol{x}+\mathbf{6})(\boldsymbol{x}-\mathbf{2})(\boldsymbol{x}+\mathbf{3})}$$
$$= \frac{\boldsymbol{A}}{(\boldsymbol{x}+\mathbf{6})} + \frac{\boldsymbol{B}}{(\boldsymbol{x}-\mathbf{2})} + \frac{\boldsymbol{C}}{(\boldsymbol{x}+\mathbf{3})}$$

A single fraction with a repeated linear factor in the denominator can be split into two or more separate fractions:

$$\frac{\mathbf{5}}{(\boldsymbol{x}+\mathbf{2})(\boldsymbol{x}-\mathbf{1})^2} = \frac{\boldsymbol{A}}{(\boldsymbol{x}+\mathbf{2})} + \frac{\boldsymbol{B}}{(\boldsymbol{x}-\mathbf{1})} + \frac{\boldsymbol{C}}{(\boldsymbol{x}-\mathbf{1})^2}$$

Improper Fractions

An **improper fraction** is one whose numerator has a degree equal to or larger than the denominator. An improper fraction must be converted to a mixed fraction before you can express it in partial fractions

e.g. $\frac{x^2+5x+8}{x-2}$ and $\frac{x^3+5x-9}{x^3-4x^2+7x-3}$ are both improper fractions

To convert an improper fraction to a mixed number

Method 1

Use algebraic long division to show that:

$$\underbrace{\frac{x^2+5x+8}{x-2}}_{F(x)} \equiv \underbrace{x+7}_{Q(x)} + \frac{\overbrace{22}^{\text{remainder}}}{\underbrace{x-2}_{\text{divisor}}}$$

Method 2

Multiply by $(x-2)$ and compare coefficients to show that:

$$\underbrace{x^2+5x+8}_{F(x)} \equiv \underbrace{(x+7)}_{Q(x)}\underbrace{(x-2)}_{\text{divisor}} + \overbrace{22}^{\text{remainder}}$$

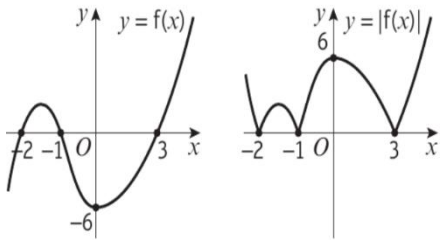
KNOWLEDGE ORGANISER: Year 13 Core – Chapter 2 – Graphs & Functions

A **modulus function** is, in general, a function of the type $y = f(x)$

- When $f(x) \geq 0, |f(x)| = f(x)$
- When $f(x) \leq 0, |f(x)| = -f(x)$

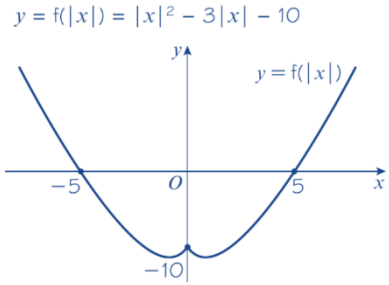
To sketch the graph of $y = |f(x)|$

- Sketch the graph of $y = f(x)$
- Reflect any parts where $f(x) < 0$ in the x-axis
- Delete the parts below the x-axis

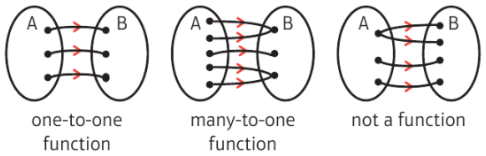


To sketch the graph of $y = f(|x|)$

- Sketch the graph of $y = f(x)$ for $x \geq 0$
- Reflect this in the y-axis

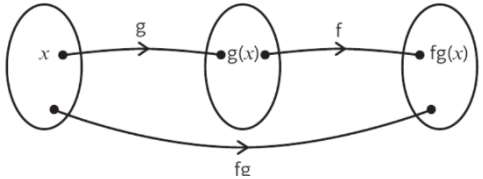


A mapping is a **function** if every input has a distinct output. Functions can either be **one-to-one** or **many-to-one**



Two or more functions can be combined to make a new function. The new function is called a **composite function**

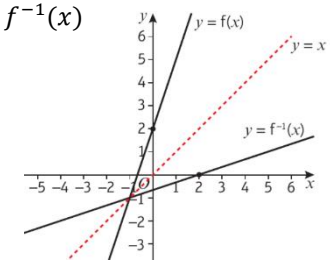
- $\boldsymbol{fg(x)}$ **means apply** $\boldsymbol{g}$ **first, then apply** $\boldsymbol{f}$
- $\boldsymbol{fg(x) = f(g(x))}$



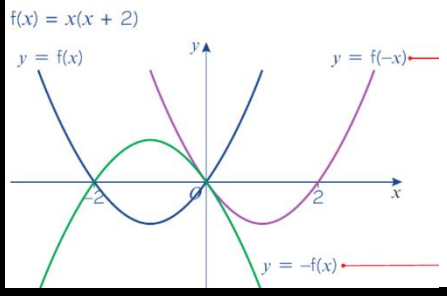
The **inverse** of a function performs the opposite operation to the original function

- **Functions** $\boldsymbol{f(x)}$ **and** $\boldsymbol{f^{-1}(x)}$ **are the inverses of each other.** $\boldsymbol{ff^{-1}(x) = f^{-1}f(x) = x}$
- **The graphs of** $\boldsymbol{f(x)}$ **and** $\boldsymbol{f^{-1}(x)}$ **are reflections of each other in the line** $\boldsymbol{y = x}$
- **The domain of** $\boldsymbol{f(x)}$ **is the range** $\boldsymbol{f^{-1}(x)}$
- **The range of** $\boldsymbol{f(x)}$ **is the domain** $\boldsymbol{f^{-1}(x)}$

the inverse $f(x)$ is written as $f^{-1}(x)$



- Multiplying by -1 '**inside**' the function **reflects** the graph in the **y-axis**



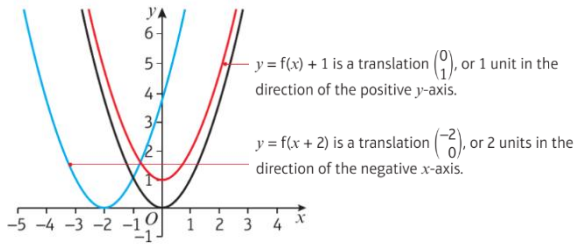
- Multiplying by -1 '**outside**' the function **reflects** the graph in the **x-axis**

$y = f(-x)$ is $y = (-x)(-x + 2)$ which is $y = x^2 - 2x$ and this is a reflection of the original curve in the y-axis. Alternatively multiply each x-coordinate by -1 and leave the y -coordinates unchanged.

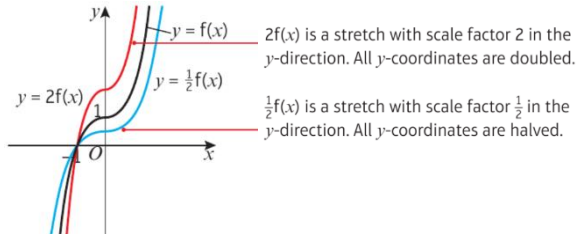
$y = -f(x)$ is $y = -x(x + 2)$ which is $y = -x^2 + 2x$ and this is a reflection of the original curve in the x-axis. Alternatively multiply each y-coordinate by -1 and leave the x -coordinates unchanged

Adding or subtracting a constant '**outside**' the function **translates** a graph **vertically.**

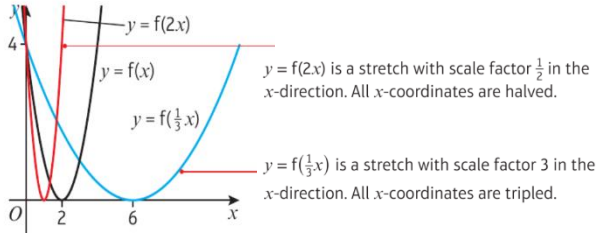
Adding or subtracting a constant '**inside**' the function **translates** a graph **horizontally.**



Multiplying by a constant '**outside**' the function **stretches** a graph in the **vertical** direction



Multiplying by a constant '**inside**' the function **stretches** a graph in the **horizontal** direction



KNOWLEDGE ORGANISER: Year 13 Core – Chapter 3 Sequences & Series

In an **Arithmetic Sequence,** the difference between consecutive terms is constant

The formula for the nth term of an arithmetic sequence is

$$\boldsymbol{u_n = a + (n-1)d}$$

where a is the first term and d is the common difference.

An arithmetic series is the sum of the terms of an arithmetic sequence.

The sum of the first $\boldsymbol{n}$ terms of an arithmetic series is given by

$$\boldsymbol{S_n = \frac{n}{2}(2a + (n-1)d),}$$

where a is the first term and d is the common difference.

You can also write this formula as

$$\boldsymbol{S_n = \frac{n}{2}(a + l),}$$

where l is the last term

A **geometric sequence** has a **common ratio** between consecutive terms

The formula for the nth term of a geometric sequence is

$$\boldsymbol{u_n = ar^{n-1}}$$

Where a is the first term and r is the common ratio

A geometric series is the sum of the terms of a geometric sequence.

The sum of the first $\boldsymbol{n}$ terms of a geometric series is given by

$$\boldsymbol{S_n = \frac{a(1-r^n)}{1-r}, r \neq 1} \quad \text{or} \quad \boldsymbol{S_n = \frac{a(r^n-1)}{r-1}, r \neq 1}$$

where a is the first term and r is the common ratio

A geometric series is **convergent** if and only if $|r| < 1$, where r is the common ratio.

The **sum to infinity** of a convergent geometric series is given by

$$\boldsymbol{S_\infty = \frac{a}{1-r}}$$

The Greek letter 'sigma' is used to signify a sum. You write it as Σ. You write the limits on the top and bottom to show which terms you are summing

This tells you that are summing the expression in brackets with $r = 1, r = 2, \ldots$ up to $r = 5$.

$$\sum_{r=1}^{5}(2r-3) = -1 + 1 + 3 + 5 + 7$$

Look at the limits carefully: they don't have to start at 1.

$$\sum_{r=3}^{7}(5 \times 2^r) = 40 + 80 + 160 + 320 + 640$$

A recurrence relation of the form $\boldsymbol{u_{n+1} = f(u_n)}$ defines each term of a sequence as a function of the previous term

Example

$u_{n+1} = u_n + 4, u_1 = 7$
Substituting $n = 1$, $u_2 = u_1 + 4 = 7 + 4 = 11$.
Substituting $n = 2$, $u_3 = u_2 + 4 = 11 + 4 = 15$.
Substituting $n = 3$, $u_4 = u_3 + 4 = 15 + 4 = 19$.
Sequence is 7, 11, 15, 19, …

- A sequence is **increasing** if $\boldsymbol{u_{n+1} > u_n}$ **for all** $\boldsymbol{n \epsilon \mathbb{N}}$
- A sequence is **decreasing** if $\boldsymbol{u_{n+1} < u_n}$ **for all** $\boldsymbol{n \epsilon \mathbb{N}}$
- A sequence is **periodic** if the terms **repeat** in a cycle. For a **periodic sequence** there is an integer $\boldsymbol{k}$ such that $\boldsymbol{u_{k+1} < u_k}$ **for all** $\boldsymbol{n \epsilon \mathbb{N}}$**.** The **value** $\boldsymbol{k}$ is called the **order of the sequence**

Examples:

- 2,3,4,5, …is an increasing sequence
- -3, -6, -12, -24, …is a decreasing sequence
- -2, 1, -2, 1, … is a periodic sequence with period 2
- 1, -2, 3, -4, 5, -6 … is not increasing, decreasing or periodic

You can **model** real-life situations with series. For example if a person's salary increases by the same percentage every year, their salaries each year would form a **geometric sequence** and the amount they have been paid in total over $\boldsymbol{n}$ years would be modelled by the corresponding **geometric sequence**

KNOWLEDGE ORGANISER: Year 13 Core – Chapter 4 – Binomial Expansion

The binomial expansion is:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n \qquad (n \in \mathbb{N})$$

This means that n must be a member of the **natural numbers**. This is all the **positive integers**

There are $n+1$ terms, so this formula produces a **finite** number of terms

If n is a **fraction** or a **negative number** you need to use a different version of the binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \cdots$$

The expansion is valid when $|x| < 1,\ n \in \mathbb{R}$

When n **is not natural number**, none of the factors in the expression $n(n-1)\ldots(n-r+1)$ are equal to zero. This means that this version of the binomial expansion produces **an infinite number of terms**

Example

$$\frac{1}{1+x} = (1+x)^{-1}$$
$$= 1 + (-1)x + \frac{(-1)(-2)x^2}{2!} + \frac{(-1)(-2)(-3)x^3}{3!} + \ldots$$
$$= 1 - 1x + 1x^2 - 1x^3 + \ldots$$
$$= 1 - x + x^2 - x^3 + \ldots$$

The binomial expansion of $(1+x)^n$ can be used to expand $(a+bx)^n$ for any constant a and b.
You need to take a factor a^n out of the expression.

$$(a+bx)^n = \left(a\left(1+\frac{bx}{a}\right)\right)^n = a^n\left(1+\frac{bx}{a}\right)^n$$

Make sure you multiply **every term** in the expansion $\left(1+\frac{bx}{a}\right)^n$ **by a^n**

Notation:

'x is small' means we can assume the expansion is valid for the x values being considered, as high powers become insignificant to the first few terms

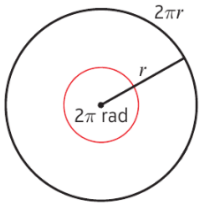
Links:

- You need to be able to express terms in index form using fractional and negative powers
- You need to be confident expressing algebraic fractions as sums of partial fractions
- You need to be confident with algebraic division

KNOWLEDGE ORGANISER: Year 13 Core – Chapter 5 – Radians

You can measure angles in degrees and radians.
The circumference of a circle of radius r is an arc length $2\pi r$, so it subtends an angle of 2π radians at the centre of the circle

$$2\pi = 360°$$
$$\pi = 180°$$
$$1\ radian = \frac{180°}{\pi}$$



NOTE: you always use radians when you are differentiating or integrating trigonometric functions.

You should learn these important angles in radians:

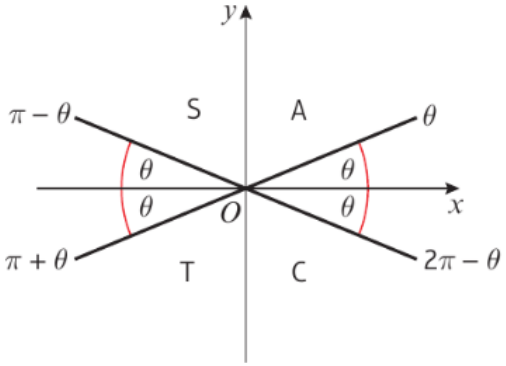
$30° = \frac{\pi}{6}$ radians
$45° = \frac{\pi}{4}$ radians
$60° = \frac{\pi}{3}$ radians
$90° = \frac{\pi}{2}$ radians
$180° = \pi$ radians
$360° = \pi$ radians

You need to learn the exact values of the trigonometric ratios of these angles measured in radians:

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

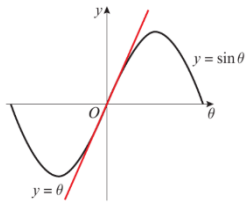
You can use these rules to find sin, cos or tan of any positive or negative angle using the corresponding acute angle made with the x-axis θ

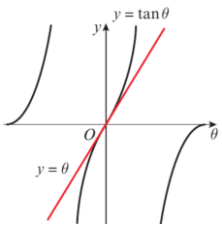
- $\sin(\pi - \theta) = \sin\theta$
- $\sin(\pi + \theta) = -\sin\theta$
- $\sin(2\pi - \theta) = -\sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos(\pi + \theta) = -\cos\theta$
- $\cos(2\pi - \theta) = \cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$
- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$

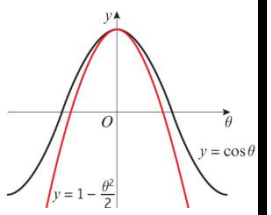


When $\boldsymbol{\theta}$ **is small and measured in radians:**

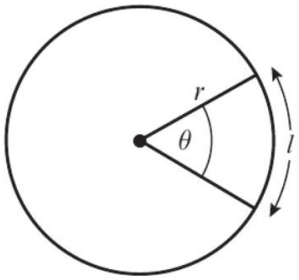
- $\sin\boldsymbol{\theta} \approx \boldsymbol{\theta}$
- $\tan\boldsymbol{\theta} \approx \boldsymbol{\theta}$
- $\cos\boldsymbol{\theta} \approx \mathbf{1} - \frac{\boldsymbol{\theta}^2}{\mathbf{2}}$



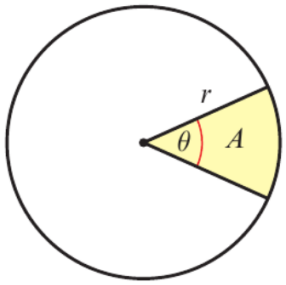




To find the **arc length** l of a sector of a circle use the formula $\boldsymbol{l = r\theta}$, where r is the radius of the circle and θ is the angle, in radians, contained by of radius the sector.



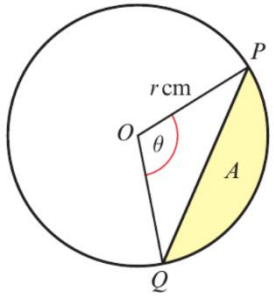
To find **area of a sector** of a circle use the formula $\boldsymbol{A = \frac{1}{2}r^2\theta}$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector



The **area of a segment** in a circle of radius r equals area of sector minus area of triangle.

$$\boldsymbol{A = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta}$$

$$\boldsymbol{A = \frac{1}{2}r^2(\theta - \sin\theta)}$$

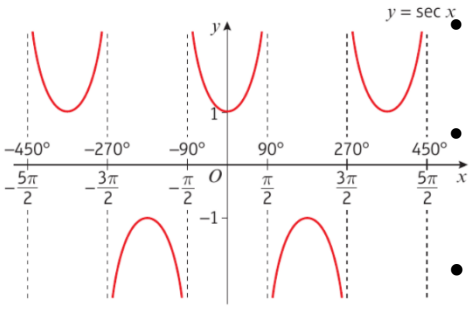


KNOWLEDGE ORGANISER: Year 13 Core – Chapter 6 – Trigonometric Functions

$$\sec x = \frac{1}{\cos x}$$

Undefined for values of x for which $\cos x = 0$

The graph of $y = \sec x,\ x \in \mathbb{R}$, has symmetry in the y-axis and has a period 360° or 2π radians. It has vertical asymptotes all the values of x for which $\cos x = 0$

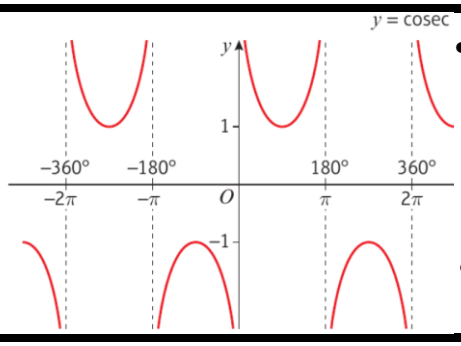


- The domain is $x \in \mathbb{R}$, $x \neq 90°, 270°$ or any odd multiple of 90°
- In radians the domain is $x \in \mathbb{R},\ x \neq \frac{\pi}{2}, \frac{3\pi}{2}$ or any odd multiple of $\frac{\pi}{2}$
- The range is $y \leq -1$ or $y \geq 1$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

Undefined for values of x for which $\sin x = 0$

The graph of $y = \operatorname{cosec} x$, $x \in \mathbb{R}$, has a period 360° or 2π radians. It has vertical asymptotes all the values of x for which $\sin x = 0$

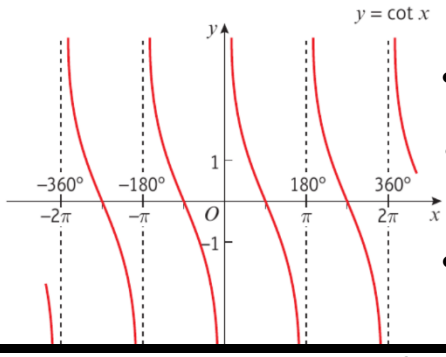


- The domain is $x \in \mathbb{R}$, $x \neq 180°, 360°$ or any multiple of 180°
- In radians the domain is $x \in \mathbb{R},\ x \neq \pi, 2\pi$ or any multiple of π
- The range is $y \leq -1$ or $y \geq 1$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

Undefined for values of x for which $\tan x = 0$

The graph of $y = \cot x,\ x \in \mathbb{R}$, has a period 180° or π radians. It has vertical asymptotes all the values of x for which $\tan x = 0$



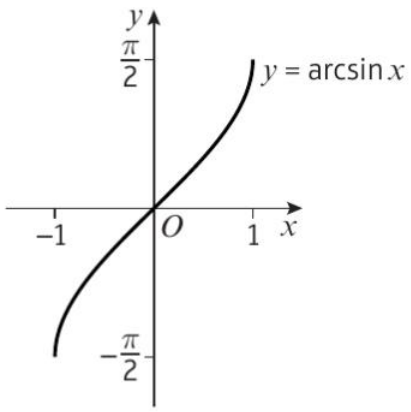
- The domain is $x \in \mathbb{R}$, $x \neq$ any multiple of 180°
- In radians the domain is $x \in \mathbb{R},\ x \neq \pi, 2\pi$ or any multiple of π
- The range is $y \in \mathbb{R}$

$\sec x = k$ and $\operatorname{cosec} x = k$ have **no solutions** for $\boldsymbol{-1 \leq k \leq 1}$

You can use the identity $\sin^2 x + \cos^2 x \equiv 1$ to prove the following identities

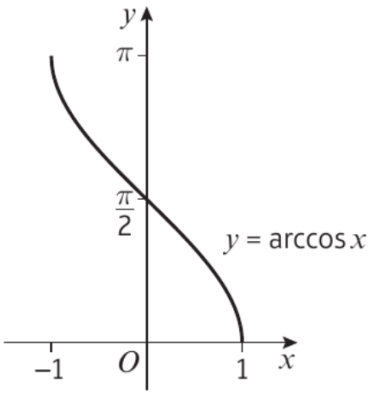
- $1 + \tan^2 x \equiv \sec^2 x$
- $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

The **inverse** function of $\sin x$ is called $\boldsymbol{\arcsin x}$



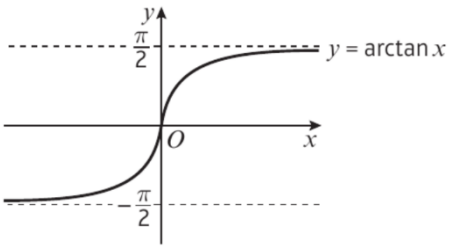
- **The domain is $\boldsymbol{-1 \leq x \leq 1}$**
- **The range is $\boldsymbol{-\frac{\pi}{2} \leq arcsinx \leq \frac{\pi}{2}}$ or $\boldsymbol{-90° \leq arcsinx \leq 90°}$**

The **inverse** function of $\cos x$ is called $\boldsymbol{\arccos x}$



- **The domain is $\boldsymbol{-1 \leq x \leq 1}$**
- **The range is $\boldsymbol{0 \leq arcsinx \leq \pi}$ or $\boldsymbol{0° \leq arcsinx \leq 180°}$**

The **inverse** function of $\tan x$ is called $\boldsymbol{\arctan x}$



- **The domain is $\boldsymbol{x \in \mathbb{R}}$**
- **The range is $\boldsymbol{-\frac{\pi}{2} \leq arctanx \leq \frac{\pi}{2}}$ or $\boldsymbol{-90° \leq arctanx \leq 90°}$**

KNOWLEDGE ORGANISER: Year 13 Core – Chapter 7 – Trigonometry & Modelling

The **addition formulae** (sometimes called the compound-angle formulae) for sine, cosine and tangent are defined as follows

- $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$
- $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$
- $\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$
- $\sin(A-B) \equiv \sin A \cos B - \cos A \sin B$
- $\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$
- $\tan(A+B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$

The **addition formulae** can be used to find exact values of trigonometric functions of different angles

Example

$$\begin{aligned}\sin 15° &= \sin(45° - 30°)\\ &= \sin 45° \cos 30° - \cos 45° \sin 30°\\ &= \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right) - \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\right)\\ &= \tfrac{1}{4}\left(\sqrt{3}\sqrt{2} - \sqrt{2}\right)\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Double-angle formulae

- $\sin 2A \equiv 2 \sin A \cos B$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$

The **double-angle formulae** can be used to eliminate variables

Example

The equations can be written as

$$\sin\theta = \frac{x}{3} \quad \cos 2\theta = \frac{3-y}{4}$$

As $\cos 2\theta \equiv 1 - 2\sin^2\theta$ for all values of θ,

$$\frac{3-y}{4} = 1 - 2\left(\frac{x}{3}\right)^2$$

So $\quad \frac{y}{4} = 2\left(\frac{x}{3}\right)^2 - \frac{1}{4}$

or $\quad y = 8\left(\frac{x}{3}\right)^2 - 1$

You can use the addition formulae to simplify some trigonometric expressions:

For positive values of a and b,

- $\boldsymbol{a\sin x + b\cos x}$ can be expressed in the form $\boldsymbol{R\sin(x+\alpha)}$
- $\boldsymbol{a\sin x - b\cos x}$ can be expressed in the form $\boldsymbol{R\sin(x-\alpha)}$
- $\boldsymbol{a\cos x + b\sin x}$ can be expressed in the form $\boldsymbol{R\cos(x-\alpha)}$
- $\boldsymbol{a\cos x - b\sin x}$ can be expressed in the form $\boldsymbol{R\cos(x+\alpha)}$

With $R > 0$ and $0 < \alpha < 90°$ (or $\frac{\pi}{2}$) and where $R\cos\alpha = a$ and $R\sin\alpha = b$ and $R = \sqrt{a^2 + b^2}$

Method: Use the addition formulae to expand $\sin(x \pm \alpha)$ or $\cos(x \mp \alpha)$ then equate coefficients

You can use trigonometric functions to model real-life situations. In the trigonometrical modelling questions, you will often have to write the model using $R\sin(x \pm \alpha)$ or $R\cos(x \mp \alpha)$ to find maximum or minimum values.

Example: Solve $2\cos\theta + 5\sin\theta$

Set $2\cos\theta + 5\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

So $R\cos\alpha = 2$ and $R\sin\alpha = 5$

Dividing, $\tan\alpha = \frac{5}{2}$, so $\alpha = 68.2°$

Squaring and adding: $R = \sqrt{29}$

So $2\cos\theta + 5\sin\theta \equiv \sqrt{29}\cos(\theta - 68.2$

$\sqrt{29}\cos(\theta - 68.2°) = 3$

So $\cos(\theta - 68.2°) = \frac{3}{\sqrt{29}}$

$\cos^{-1}\left(\frac{3}{\sqrt{29}}\right) = 56.1...°$

So $\theta - 68.2° = -56.1...°, 56.1...°$

$\theta = 12.1°, 124.3°$ (to the nearest 0.1°)

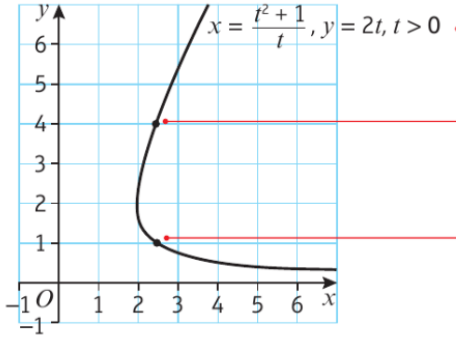
KNOWLEDGE ORGANISER: Year 13 Core – Chapter 8 – Parametric Equations

You can write the x- and y-coordinates of each point on a curve as functions of a third variable. This variable is called a parameter and is often represented by the letter t.

- **A curve can be defined using parametric equations $x = p(t)$ and $y = q(t)$. Each value of the parameter, t defines a point on the curve with coordinates $(p(t), q(t))$.**

BEWARE:

- The value of the parameter t is generally not equal to either the x- or y-coordinate
- More than one point on the curve can have the same x-coordinate



These are the parametric equations of the curve. The domain of the parameter tells you the values of t you would need to substitute to find the coordinates of the points on the curve.

When $t = 2$, $x = \frac{2^2+1}{2} = 2.5$ and $y = 2 \times 2 = 4$.

This corresponds to the point (2.5, 4).

When $t = 0.5$, $x = \frac{0.5^2+1}{0.5} = 2.5$ and $y = 2 \times 0.5 = 1$.

This corresponds to the point (2.5, 1).

For parametric equations $x = p(t)$ and $y = q(t)$ with Cartesian equation $y = f(x)$:

- **The domain of $f(x)$ is the range of $p(t)$**
- **The range of $f(x)$ is the range of $q(t)$**

You can convert between parametric equations and Cartesian equations by using substitution to eliminate the parameter.

Example 1:

$x = 2t$ so $t = \frac{x}{2}$ (1)

$y = t^2$ (2)

Substitute (1) into (2):

$y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$

Example 2:

$x = \sin t + 2$

So $\sin t = x - 2$ (1)

$y = \cos t - 3$

$\cos t = y + 3$ (2)

Substitute (1) and (2) into

$\sin^2 t + \cos^2 t \equiv 1$

$(x - 2)^2 + (y + 3)^2 = 1$

Modelling:

- You can use parametric equations to model real-life situations.
- If you have to comment on a modelling assumption or range of validity, consider whether the assumption is realistic given the context of the question. Make sure you refer to the real-life situation being modelled in your answer.

KNOWLEDGE ORGANISER: Year 13 Core – Chapter 9 – Differentiation

The Chain rule is used to differentiate composite functions or functions of functions

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Where y is a function of u and u is another function of x

Exponential and Logarithm Differentials

- If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$
- If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$
- If $y = a^{kx}$ then $\frac{dy}{dx} = a^{kx}k \ln a$
- If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$
- If $y = \ln x(f(x))$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

Some equations are difficult to rearrange into the form $y = f(x)$ or $x = f(y)$. You can sometimes differentiate these equations **implicitly** without rearranging them.

- $\frac{d}{dx}(f(y)) = f'(y)\frac{dy}{dx}$
- $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$

To **differentiate** functions which are **express as x in term of y** use

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

The Product Rule is used to differentiate a product of two functions:

If $y = uv$ then, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Where u and v are functions of x

Trigonometric Differentials

- If $y = \sin kx$, then $\frac{dy}{dx} = k \cos kx$
- If $y = \cos kx$, then $\frac{dy}{dx} = -k \sin kx$
- If $y = \tan kx$, then $\frac{dy}{dx} = k \sec^2 kx$
- If $y = \operatorname{cosec} kx$, then $\frac{dy}{dx} = -k \operatorname{cosec} kx \cot kx$
- If $y = \sec kx$, then $\frac{dy}{dx} = -k \sec kx \tan kx$
- If $y = \cot kx$, then $\frac{dy}{dx} = -k \operatorname{cosec}^2 kx$

You can use the second derivative to determine whether a curve is **concave** or **convex** on a given domain

- The function $\boldsymbol{f(x)}$ **is concave** on a given interval if and only if $\boldsymbol{f''(x) \leq 0}$ for every value of x in that interval
- The function $\boldsymbol{f(x)}$ **is convex** on a given interval if and only if $\boldsymbol{f''(x) \geq 0}$ for every value of x in that interval

LINKS: to find the second derivative, $f''(x)$ or $\frac{d^2y}{dx^2}$ you differentiate twice with respect to x

The Quotient Rule is used to differentiate a quotient of two functions i.e. a function divided by a function

If $y = \frac{u}{v}$ then, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Parametric Differentiation – when functions are defined parametrically, you can find the gradient at a given point without converting into Cartesian form

If x and y are given as functions of a parameter, t:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

A **point of inflection** is a point at which $f''(x)$ or $\frac{dy}{dx}$ changes sign

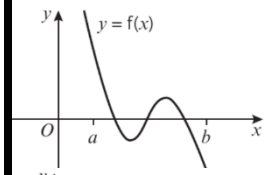
The **Chain Rule** can be used **to connect rates of change** in situations **involving more than two variables**

KNOWLEDGE ORGANISER: Year 13 Core – Chapter 10 – Numerical Methods

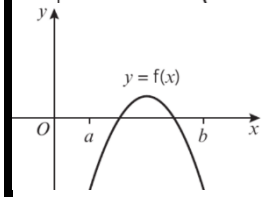
A **root of a function is a value of** x for which $f(x) = 0$. The graph of $y = f(x)$ will cross the x-axis at points corresponding to the roots of the function.
You can sometimes show that a root exists within a given interval by showing that the function changes sign within the interval

- **If the function $f(x)$ is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ has at least one root, x, which satisfies $a < x < b$**

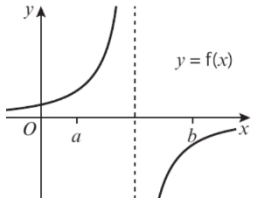
BEWARE: A change of sign does not necessarily mean there is exactly one root, and the absence of a sign change does not necessarily mean that a root does not exist in the interval.



There are multiple roots within the interval $[a, b]$. In this case there is an **odd number** of roots



There are multiple roots within the interval $[a, b]$, but a sign change does not occur. In this case there is an **even number** of roots



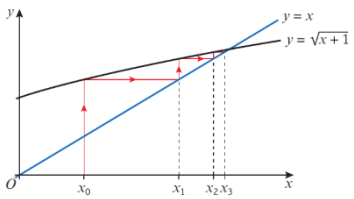
There is a vertical asymptote within the interval $[a, b]$. A sign change does occur, but there is no root.

Iteration is a method used to find a value of x for which $f(x) = 0$.

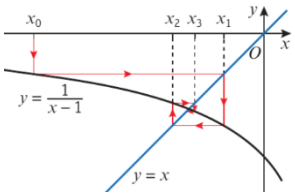
To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into the form $x = g(x)$ and use the iterative formula
$x_{n+1} = g(x_n)$

Some Iterations **converge** to a root. This can happen in two ways

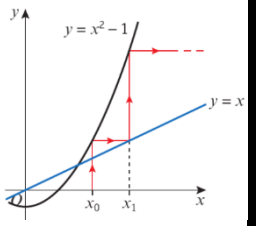
1. Successive iterations get closer and closer to the root from the same direction and this results in a **Staircase diagram**



2. Successive iterations alternate being above the below the root and this results in a **Cobweb diagram**



When an iteration moves away from a root it **diverges** as shown

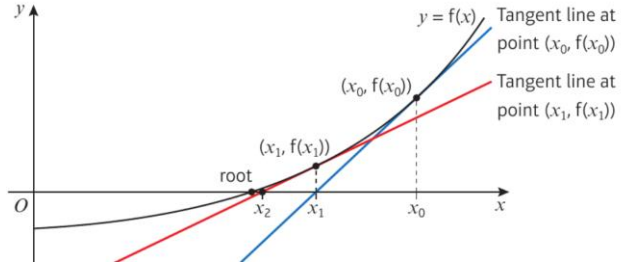


The Newton-Raphson Method is used to find numerical solutions to equations of the form $f(x) = 0$. You need to be able to differentiate $f(x)$ to use this method

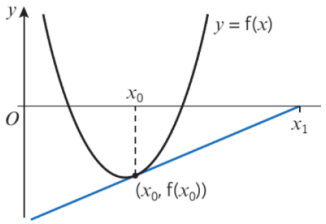
The Newton-Raphson Formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

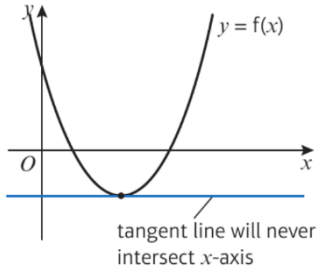
The method uses tangent lines to find increasingly accurate approximations of a root. The value of x_{n+1} is the point at which the tangent to the graph at $(x_n, f(x_n))$ intersects the x-axis.



BEWARE: If the starting value is not chosen carefully, the method can converge on a root very slowly or can fail completely



Because x_0 is close to a turning point the gradient of the tangent at $(x_0, f(x_0))$ is small, so it intercepts the x-axis a long way from x_0.



KNOWLEDGE ORGANISER: Year 13 Core – Chapter 11 – Integration

Integrating Standard Functions

- $\int kx^n dx = \frac{k}{n+1}x^{n+1} + c \qquad n \neq 0$
- $\int e^x dx = e^x + c$
- $\int \frac{1}{x} dx = ln\,|x| + c$
- $\int \cos x\, dx = \sin x + c$
- $\int \sin x\, dx = -\cos x + c$
- $\int \sec^2 x\, dx = \tan x + c$
- $\int \text{cosec}\, x\, dx = -\text{cosec}\, x + c$
- $\int \text{cosec}^2 x\, dx = -\cot x + c$
- $\int \sec x \tan x\, dx = \sec x + c$

Trigonometric identities can be used to integrate expressions. This allows an expression that cannot be integrated to be replaced by an identical expression that can be integrated.

<u>Example</u> – find $\int \tan^2 x\, dx$

Since $\sec^2 x \equiv 1 + \tan^2 x$

$\tan^2 x \equiv \sec^2 x - 1$

So $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$

$= \int \sec^2 x\, dx - \int 1\, dx$

$= \tan x - x + c$

You cannot integrate $\tan^2 x$ but you can integrate $\sec^2 x$ directly

Integration by Parts

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

- You need to write the function you are integrating in the form $u\frac{dv}{dx}$
- You will have to choose what to set as u and what to set as $\frac{dv}{dx}$
- For expression like $x\cos x$, $x^2 \sin x$ and $x^3 e^x$ let u equal the x^n term.
- When the expression involves $\ln x$ let u equal the $\ln x$ term.
- When applying limits, apply them to the uv and the $\int v\frac{du}{dx}dx$ separately

Integrating $f(ax+b)$

If you know the integral of a function, $f(x)$ you can integrate a function of the form $f(ax+b)$ using the reverse of the chain rule for differentiation. In general:

$$\int f'(ax+b)dx = \frac{1}{a}f(ax+b) + c$$

Sometimes you can simplify an integral by changing the variable. The process is called **Integration by Substitution**

- To evaluate a definite integral you have to be careful of whether your limits are x values or u values.

Using **partial fractions** enables an expression to be transformed into two or more expressions that are easier to integrate.

<u>Example</u>

$$\int \frac{x-5}{(x+1)(x-2)}dx = \int \left(\frac{2}{x+1} - \frac{1}{x-2}\right)dx$$

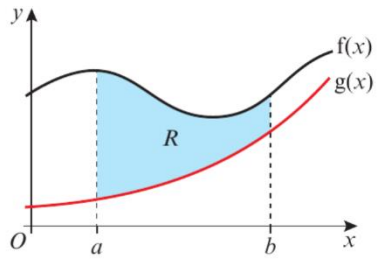
- **Integrating** expressions of the form $\int k\frac{f'(x)}{f(x)}dx$, **try** $ln|f(x)|$ and differentiate to check, and then adjust any constant

- **Integrating** an expression of the form $\int kf'(x)(f(x))^n dx$, **try** $(f(x))^{n+1}$ and differentiate to check, and then adjust any constant

The area bounded by two curves can be found using integration:

Area R

$= \int_a^b (f(x) - g(x))\, dx = \int_a^b (f(x)\, dx - \int_a^b g(x)\, dx$



Differential Equations:

When $\frac{dy}{dx} = f(x)g(y)$ you can rearrange it to give

$$\int \frac{1}{g(y)}dy = \int f(x)dx$$

The Trapezium rule:

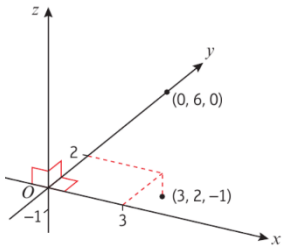
$$\int_a^b y\, dx \approx \frac{h}{2}(y_0 + 2(y_1 + y_2 \ldots + y_{n-1}) + y_n)$$

Where $h = \frac{b-a}{n}$ **a**

KNOWLEDGE ORGANISER: Year 13 Core – Chapter 12 – Vectors

The **coordinates** of a point in **3-D** are written as $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$

You can use Pythagoras' theorem in 3D to find distances on a 3D coordinate grid



The distance from the origin to the point (x, y, z) is

$$\sqrt{x^2 + y^2 + z^2}$$

The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Solving Geometric Problems

You need to be able to solve geometric problems involving vectors in three dimensions

BEWARE: AB refers to a **line segment** between A and B (or its length), whereas $\overrightarrow{AB}$ refers to the **vector** from A to B.

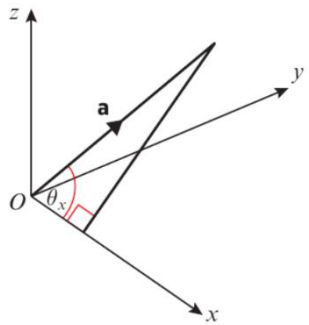
Note that $AB = BA$, $|AB| = |BA|$ but $\overrightarrow{AB} \neq \overrightarrow{BA}$

- **You can use 3D vectors to describe position and displacement relative to the $\boldsymbol{x}$-, $\boldsymbol{y}$- and $\boldsymbol{z}$-axes.**
- **You can represent 3D vectors as column vectors or using the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and k**
- **The unit vectors along $\boldsymbol{x}$-, $\boldsymbol{y}$- and $\boldsymbol{z}$-axes are denoted by $\boldsymbol{i}$, $\boldsymbol{j}$ and k respectively.**

$$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- For any 3D vector $p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

You can find the angle between a given vector and any of the coordinate axes by considering the appropriate right-angled triangle.



If the vector $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ makes an angle $\boldsymbol{\theta}$ with the positive $\boldsymbol{x}$-axis then

$$\cos \boldsymbol{\theta}_{\boldsymbol{x}} = \frac{\boldsymbol{x}}{|\boldsymbol{a}|}$$

And similarly for the angles $\boldsymbol{\theta}_{\boldsymbol{y}}$ and $\boldsymbol{\theta}_{\boldsymbol{z}}$

To add three column vectors, add the x-components, the y-components and the z-components

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} p + u \\ q + v \\ r + w \end{pmatrix}$$

To multiply a column vector by a scalar, multiply each component by the scalar:

$$\lambda \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \lambda p \\ \lambda q \\ \lambda r \end{pmatrix}$$

Coplanar vectors are vectors which are in the same plane.

Non-coplanar vectors are vectors which are not in the same plane.

If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors in three dimensions which do not all lie on the same plane then you can compare their coefficients on both sides of an equation

Since, the vectors $\boldsymbol{i}, \boldsymbol{j}$ and **k** are non-coplanar,

if $p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$

then $p = u, q = v$ and $r = w$