June 2018 P1 – Solutions

1/ $$\frac{1-\cos 4\theta}{2\theta \sin 3\theta} \approx \frac{1-\left(1-\frac{(4\theta)^2}{2}\right)}{2\theta(3\theta)}$$

$$= \frac{1-1+\frac{16\theta^2}{2}}{6\theta^2}$$

$$= \frac{8\theta^2}{6\theta^2} = \frac{8}{6} = \underline{\underline{\frac{4}{3}}}$$

2/ $y = x^2 - 2x - 24x^{\frac{1}{2}}$

(a) (i) $\frac{dy}{dx} = \underline{\underline{2x - 2 - 12x^{-\frac{1}{2}}}}$

(ii) $\frac{d^2y}{dx^2} = \underline{\underline{2 + 6x^{-\frac{3}{2}}}}$

(b) $x = 4$, $\frac{dy}{dx} = 2(4) - 2 - 12(4)^{-\frac{1}{2}} = 0$

$\therefore$ a stationary point when $x = 4$.

(c) $x = 4$, $\frac{d^2y}{dx^2} = 2 + 6(4)^{-\frac{3}{2}} = 2.75$

$\frac{d^2y}{dx^2} > 0$, so this is a <u>minimum point</u>.

3, $\frac{1}{2} r^2 \theta = 11$ ①

$$\text{perimeter} = 4 \times \text{arc length}$$
$$r\theta + 2r = 4r\theta$$
$$2r = 3r\theta$$
$$2 = 3\theta$$
$$\theta = \frac{2}{3}$$

using ①, $\frac{1}{2} r^2 \left(\frac{2}{3}\right) = 11$
$$\frac{1}{3} r^2 = 11$$
$$r^2 = 33$$
$$r = \underline{\underline{\sqrt{33}}}$$

[Note: "exact" and radius is positive]

4, (a) $y = 2\ln(8-x)$ & $y = x$

$$x = 2\ln(8-x)$$

Let $f(x) = 2\ln(8-x) - x$

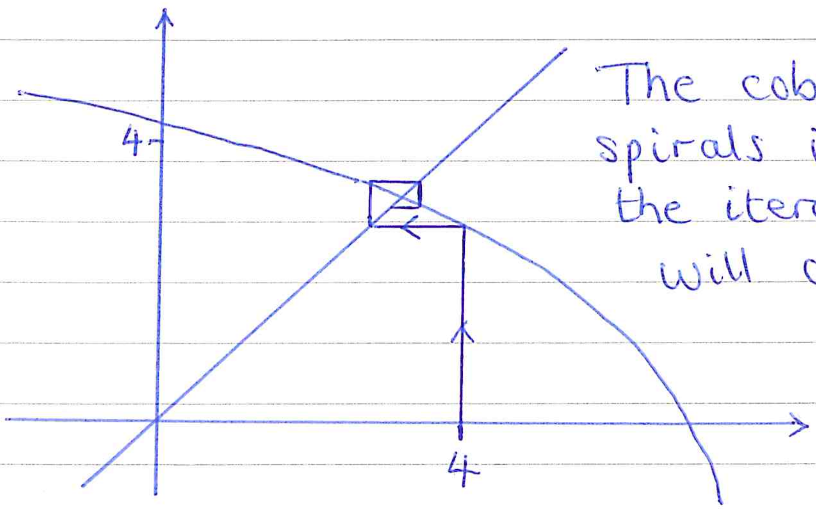
$f(3) = 2\ln(8-3) - 3 = 0.2188758249$

$f(4) = 2\ln(8-4) - 4 = -1.227411278$

Since $f(3) > 0$, $f(4) < 0$ and $f(x)$ is continuous between $x = 3$ & 4, $f(x)$ has a root in this interval.

$\therefore$ $3 < \alpha < 4$.

(b)



The cobweb diagram spirals inwards, so the iterative sequence will converge on the root.

5) $$y = \frac{3\sin\theta}{2\sin\theta + 2\cos\theta} \quad \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$$

$$\frac{dy}{d\theta} = \frac{v\frac{du}{d\theta} - u\frac{dv}{d\theta}}{v^2}$$

$$= \frac{(2\sin\theta + 2\cos\theta)(3\cos\theta) - 3\sin\theta(2\cos\theta - 2\sin\theta)}{(2\sin\theta + 2\cos\theta)^2}$$

$$= \frac{\cancel{6\sin\theta\cos\theta} + 6\cos^2\theta - \cancel{6\sin\theta\cos\theta} + 6\sin^2\theta}{4\sin^2\theta + 8\sin\theta\cos\theta + 4\cos^2\theta}$$

$$= \frac{6(\cos^2\theta + \sin^2\theta)}{4(\sin^2\theta + \cos^2\theta) + 8\sin\theta\cos\theta}$$

$$= \frac{6(1)}{4(1) + 8\sin\theta\cos\theta}$$

$$= \frac{6}{4 + 4(2\sin\theta\cos\theta)}$$

$$= \frac{6}{4 + 4\sin 2\theta}$$

$$= \frac{6}{4(1 + \sin 2\theta)}$$

$$= \frac{6/4}{1 + \sin 2\theta}$$

$$= \frac{3/2}{1 + \sin 2\theta} \qquad (A = 3/2) \qquad \square$$

6 (a) l has gradient 2

$\Rightarrow$ PA has gradient $-\frac{1}{2}$

Using $(7,5)$,

$$y - y_1 = m(x - x_1)$$
$$y - 5 = -\tfrac{1}{2}(x-7)$$
$$y - 5 = -\tfrac{1}{2}x + \tfrac{7}{2}$$
$$2y - 10 = -x + 7$$
$$2y + x = 17 \quad \square$$

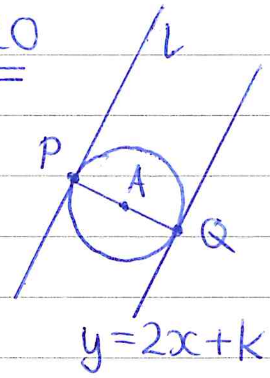
(b) P is the point of intersection,

$$2(2x+1) + x = 17$$
$$4x + 2 + x = 17$$
$$5x = 15$$
$$x = 3$$
$$y = 2x+1 = 2(3)+1 = 7$$

P is $(3,7)$

$$AP = \sqrt{(7-3)^2 + (5-7)^2}$$
$$= \sqrt{4^2 + (-2)^2}$$
$$= \sqrt{20} \qquad (= \text{radius})$$

eq^n of circle, $\underline{\underline{(x-7)^2 + (y-5)^2 = 20}}$ r^2



(c) Let PQ be a diameter

$$\overrightarrow{PA} = \begin{pmatrix} 7-3 \\ 5-7 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \overrightarrow{AQ}$$

$$Q = (7+4,\ 5-2) = (11,3)$$

Where circle meets line,

$$(x-7)^2 + ((2x+k)-5)^2 = 20$$
$$x^2 - 14x + 49 + 4x^2 + 4kx + k^2 - 10(2x+k) + 25 = 20$$
$$5x^2 - 34x + 4kx + 49 + k^2 - 10k + 25 = 20$$
$$5x^2 + (4k-34)x + (k^2 - 10k + 54) = 0$$

6/ (c) (cont.)

$b^2 - 4ac = 0$ (as line is a tangent)

$(4k-34)^2 - 4(5)(k^2-10k+54) = 0$

$16k^2 - 272k + 1156 - 20k^2 + 200k - 1080 = 0$

$-4k^2 - 72k + 76 = 0$

$-k^2 - 18k + 19 = 0$

$k^2 + 18k - 19 = 0$

$(k+19)(k-1) = 0$

$(k = 1)$ (line l)

or $\underline{\underline{k = -19}}$ (required value)

7/ (a) $\int_k^{3k} \frac{2}{3x-k}\,dx = \left[\frac{1}{3}(2)\ln(3x-k)\right]_k^{3k}$

$= \frac{2}{3}\left\{(\ln(3(3k)-k) - (\ln(3(k)-k))\right\}$

$= \frac{2}{3}\left\{\ln 8k - \ln 2k\right\}$

$= \frac{2}{3}\ln\frac{8k}{2k} = \underline{\frac{2}{3}\ln 4}$

$\therefore$ independent of k □

(b) $\int_k^{2k} 2(2x-k)^{-2}\,dx = \left[-(2x-k)^{-1}\right]_k^{2k}$

$= \left(-(2(2k)-k)^{-1}\right) - \left(-(2(k)-k)^{-1}\right)$

$= -\frac{1}{3k} + \frac{1}{k}$

$= -\frac{1}{3k} + \frac{3}{3k}$

$= \frac{2}{3k} = \frac{2}{3} \times \frac{1}{k}$ (or $\frac{2/3}{k}$)

$\therefore$ inversely proportional to k. □

8) $D = 5 + 2\sin(30t)$

(a) At 6·30am, $t = 6\cdot5$

$$D = 5 + 2\sin(30 \times 6\cdot5)$$
$$= 4\cdot48236191$$
$$= \underline{\underline{4\cdot48\text{ m}}} \quad (3sf)$$

[NB: units]

(b) $5 + 2\sin(30t) = 3\cdot8$

$$\sin(30t) = -0\cdot6$$

$[\sin^{-1} 0\cdot6 = 36\cdot86989765]$

S	A
✓ T	C ✓

$180-\theta$	θ
✓ $180+\theta$	$360-\theta$ ✓

$$180 + 36\cdot86989765 = 216\cdot8698976$$
$$360 - 36\cdot86989765 = 323\cdot1301024$$
$$30t = 216\cdot8698976,\ 323\cdot1301024$$
$$t = 7\cdot228996588,\ 10\cdot77100341$$

[before 8·30 am]

$$0\cdot77100341 \times 60 = 46\cdot26020471$$

$\underline{\underline{10:46\text{ am}}}$ (nearest min.)

9) (a) $x^2 - \underbrace{2x}_{u}\underbrace{y}_{v} + 3y^2 = 50$

$$2x - \left[2x(1)\frac{dy}{dx} + y(2)\right] + 6y\frac{dy}{dx} = 0$$
$$2x - 2x\frac{dy}{dx} - 2y + 6y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(6y - 2x) = 2y - 2x$$
$$\frac{dy}{dx} = \frac{2y-2x}{6y-2x}$$
$$= \frac{y-x}{3y-x} \quad \square$$

9, (b) At P, Q, gradient of tangent is infinite [denominator]

$$\Rightarrow \quad 3y - x = 0$$
$$x = 3y$$
$$(3y)^2 - 2(3y)y + 3y^2 = 50$$
$$9y^2 - 6y^2 + 3y^2 = 50$$
$$6y^2 = 50$$
$$y^2 = \frac{50}{6} = \frac{25}{3}$$

[y negative at P] $\quad y = -\sqrt{\frac{25}{3}} = -\frac{5}{\sqrt{3}} = -\frac{5\sqrt{3}}{3}$

$$x = 3y = 3\left(-\frac{5\sqrt{3}}{3}\right) = -5\sqrt{3}$$

$$\underline{\underline{P\left(-5\sqrt{3}, -\frac{5\sqrt{3}}{3}\right)}}$$

(c) Use $\frac{dy}{dx} = 0 \Rightarrow y - x = 0 \Rightarrow x = y$

Solve this simultaneously with $x^2 - 2xy + 3y^2 = 50$.
Select the solution where $x, y > 0$.

10/ (a) $$\frac{dH}{dt} = \frac{H\cos(0.25t)}{40}$$

$$\int \frac{40}{H}\,dH = \int \cos(0.25t)\,dt$$

$$40\ln H = 4\sin(0.25t) + c$$

Given $H=5$ when $t=0$,

$$40\ln 5 = 4\sin(0.25\times 0) + c$$

$$40\ln 5 = 4(0) + c$$

$$c = 40\ln 5$$

$$40\ln H = 4\sin(0.25t) + 40\ln 5$$

$$\ln H = 0.1\sin(0.25t) + \ln 5$$

$$\ln H - \ln 5 = 0.1\sin(0.25t)$$

$$\ln\left(\frac{H}{5}\right) = 0.1\sin(0.25t)$$

$$\frac{H}{5} = e^{0.1\sin(0.25t)}$$

$$H = 5e^{0.1\sin(0.25t)} \quad \square$$

(b) $$\max H = 5e^{0.1\times 1}$$

$$= 5e^{0.1}$$

$$= 5.52585459$$

$$= \underline{\underline{5.53\text{ m}}} \quad (3\text{ sf})$$

(c) $$\sin(0.25t) = 1 \Rightarrow 0.25t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$$

$$0.25T = \frac{5\pi}{2}$$

$$0.5T = 5\pi$$

$$T = 10\pi$$

$$= 31.41592654$$

$$= \underline{\underline{31.4\text{ secs}}} \quad (3\text{ sf})$$

11/ (a) $\sqrt{\dfrac{1+4x}{1-x}} = (1+4x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$

$$(1+4x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}(4x) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}(4x)^2 + \ldots$$
$$= 1 + 2x - 2x^2 + \ldots$$

$$(1-x)^{-\frac{1}{2}} = 1 + (-\tfrac{1}{2})(-x) + \frac{-\frac{1}{2}(-\frac{3}{2})}{2!}(-x)^2$$
$$= 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2$$

$$(1+2x-2x^2)(1+\tfrac{1}{2}x+\tfrac{3}{8}x^2)$$
$$= 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + 2x + x^2 + \tfrac{3}{4}x^3 - 2x^2 + \ldots$$
$$= \underline{1 + \tfrac{5}{2}x - \tfrac{5}{8}x^2 + \ldots} \quad \square$$

(b) Expansion of $(1+4x)^{\frac{1}{2}}$ is valid for $-1 < 4x < 1 \Rightarrow -\frac{1}{4} < x < \frac{1}{4}$

However, $\frac{1}{2} > \frac{1}{4}$, so the approximation would not be accurate.

(c) $$\sqrt{\frac{1+4x}{1-x}} \approx 1 + \tfrac{5}{2}x - \tfrac{5}{8}x^2$$

$$\sqrt{\frac{1+4(\frac{1}{11})}{1-(\frac{1}{11})}} \approx 1 + \tfrac{5}{2}(\tfrac{1}{11}) - \tfrac{5}{8}(\tfrac{1}{11})^2$$

$$\sqrt{\frac{3}{2}} \approx \frac{1183}{968}$$

$$\left[\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}\right]$$

$$\frac{\sqrt{6}}{2} \approx \frac{1183}{968}$$

$$\sqrt{6} \approx \underline{\underline{\frac{1183}{484}}}$$ [NB: "simplest form"]

12/ (a) (i) $32000 = Ap^4$ ①
$50000 = Ap^{11}$ ②

② ÷ ①, $\frac{25}{16} = p^7$

$$p = \sqrt[7]{\frac{25}{16}}$$
$$= 1.065831558$$
$$= \underline{\underline{1.0658}} \quad (4\text{dp})$$

(ii) $32000 = A(1.065831558)^4$
$$A = 24796.8022$$
$$\approx 24800 \quad \square$$

(b)(i) The car is worth approx. £24800 on 1st January 2001

(ii) The value of the car increases by approx. 6.58% per year.

(C) $24796.8022(1.065831558)^t = 100\,000$
$$1.065831558^t = 4.032778065$$
$$t \log 1.065831558 = \log 4.032778065$$
$$t = \frac{\log 4.032778065}{\log 1.065831558}$$
$$= 21.8719931$$

$2001 + 21.8719931$ years
= during $\underline{\underline{2022}}$

13/ $\int_0^2 2x\sqrt{x+2}\,dx$

Let $u = \sqrt{x+2}$

$\Rightarrow u^2 = x+2 \Rightarrow x = u^2 - 2$

$\frac{dx}{du} = 2u$

$\Rightarrow dx = 2u\,du$

$x = 2, \quad u = \sqrt{2+2} = \sqrt{4} = 2$

$x = 0, \quad u = \sqrt{2+0} = \sqrt{2}$

$\int_{\sqrt{2}}^{2} 2(u^2-2)\,u \,.\, 2u\,du$

$= \int_{\sqrt{2}}^{2} 4u^2(u^2-2)\,du$

$= \int_{\sqrt{2}}^{2} 4u^4 - 8u^2\,du$

$= \left[\frac{4}{5}u^5 - \frac{8}{3}u^3\right]_{\sqrt{2}}^{2}$

$= \left(\frac{4}{5}(2)^5 - \frac{8}{3}(2)^3\right) - \left(\frac{4}{5}(\sqrt{2})^5 - \frac{8}{3}(\sqrt{2})^3\right)$

$= \left(\frac{128}{5} - \frac{64}{3}\right) - \left(\frac{4}{5}(4\sqrt{2}) - \frac{8}{3}(2\sqrt{2})\right)$

$= \frac{64}{15} - \frac{16}{5}\sqrt{2} + \frac{16}{3}\sqrt{2}$

$= \frac{64}{15} + \frac{32}{15}\sqrt{2}$

$= \frac{32}{15}(2+\sqrt{2}) \quad \square$

[Other suitable methods include substitution with $u = x+2$ and integration by parts with $u = 2x$ and $dv/dx = \sqrt{x+2}$]

14/ (a) $x = 3 + 2\sin t \Rightarrow \sin t = \frac{1}{2}(x-3)$

$$\begin{aligned} y &= 4 + 2\cos 2t \\ &= 4 + 2[1 - 2\sin^2 t] \\ &= 6 - 4\sin^2 t \\ &= 6 - 4[\tfrac{1}{2}(x-3)]^2 \\ &= 6 - 4(\tfrac{1}{4})(x-3)^2 \\ y &= 6 - (x-3)^2 \quad \square \end{aligned}$$

(b) [$y = 6 - (x-3)^2$ is quadratic with turning point at $(3, 6)$]

(i) $0 \leqslant t < 2\pi$

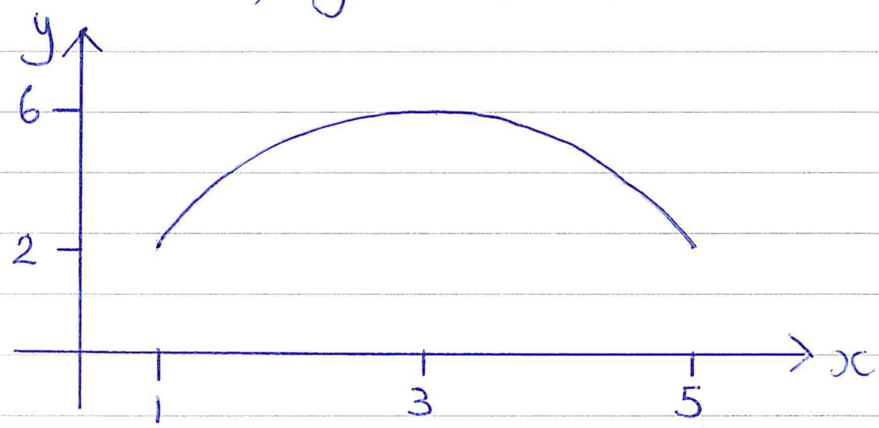
$\Rightarrow -1 \leqslant \sin t \leqslant 1$

$\Rightarrow 1 \leqslant 3 + 2\sin t \leqslant 5$

$\Rightarrow 1 \leqslant x \leqslant 5$

$x = 1,\ y = 6 - (1-3)^2 = 2$

$x = 5,\ y = 6 - (5-3)^2 = 2$



(ii) As shown above, for $0 \leqslant t < 2\pi$ it follows that $1 \leqslant x \leqslant 5$. Therefore only this section of the curve $y = 6 - (x-3)^2$ is part of C.

14, (c) $x + y = k \Rightarrow y = k - x$

$k - x = 6 - (x-3)^2$

$k - x = 6 - (x^2 - 6x + 9)$

$k - x = 6 - x^2 + 6x - 9$

$x^2 - 7x + (k+3) = 0$

$b^2 - 4ac > 0$

$(-7)^2 - 4(1)(k+3) > 0$

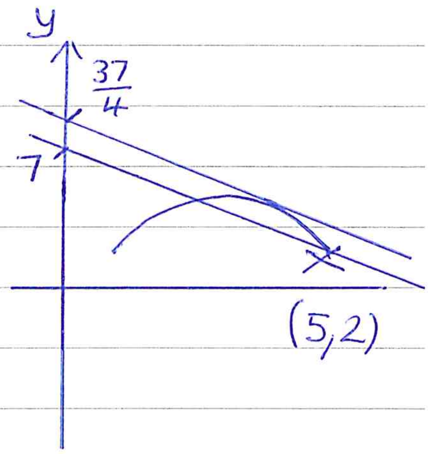
$49 - 4k - 12 > 0$

$-4k + 37 > 0$

$-4k > -37$

$4k < 37$

$k < \frac{37}{4}$



Using (5, 2),

$x + y = k$

$\Rightarrow k = 5 + 2 = 7$

$7 \leq k < \frac{37}{4}$

In set notation,

$k = \left\{ k : 7 \leq k < \frac{37}{4} \right\}$